

# ON THE $\mathbb{Z}_2$ -COHOMOLOGY OF 3D POLYHEDRAL APPROXIMATIONS

ROCIO GONZALEZ-DIAZ, JAVIER LAMAR, AND RONALD UMBLE

**ABSTRACT.** The cohomology algebra of an object encodes important geometrical information not captured by its cohomology groups. Let  $I = (\mathbb{Z}^3, 26, 6, B)$  be a 3D digital image, let  $Q(I)$  be the associated cubical complex and let  $\partial Q(I)$  be the subcomplex of  $Q(I)$  whose maximal cells are the quadrangles of  $Q(I)$  shared by a voxel of  $B$  (the foreground) and a voxel of  $\mathbb{Z}^3 \setminus B$  (the background). We show how to simplify the combinatorial structure of  $\partial Q(I)$  and obtain a polyhedral complex  $P(I)$ , which is homeomorphic to  $Q(I)$  but with fewer cells. We provide an algorithm that computes cup products on  $H^*(P(I); \mathbb{Z}_2)$  directly from the combinatorics. The computational method introduced here can be effectively applied to any polyhedral approximation of a 3D object.

## 1. INTRODUCTION

This paper completes the work proposed in our prequel-abstract [6] by providing additional insights, examples, proofs and a corrected version of formula (1) in Theorem 4.1. All modules herein are assumed to have  $\mathbb{Z}_2$  coefficients.

Let  $X$  be a cellular complex embedded in 3-dimensional space and constructed by gluing 3-dimensional polyhedra together along common faces (see [8]). At the vector space level, the cellular cohomology  $H^*(X)$  is generated by the connected components of  $X$ , homotopy classes of non-contractible loops modulo commutator, i.e.,  $\pi_1(X) / [\pi_1(X), \pi_1(X)]$ , and surfaces bounding the cavities in  $X$ . At the algebra level, the cup product encodes certain relationships among these generators, and improves our ability to distinguish between objects. For example,  $H^*(S^1 \vee S^1 \vee S^2)$  and  $H^*(S^1 \times S^1)$  are isomorphic as vector spaces but not as algebras since 2-dimensional cup products vanish in the wedge but not in the product; consequently,  $S^1 \vee S^1 \vee S^2$  and  $S^1 \times S^1$  have quite different topological properties.

To date, the cup product has seen limited application to problems in 3D image processing. In [7], Gonzalez-Diaz and Real used their 14-adjacency algorithm and the standard formulation in [13] to compute cup products on the simplicial complex  $K(I)$  associated with a given digital image  $I$ . In contrast, Kaczynski, Mischaikow and Mrozek [9] showed how to compute the homology of a digital image directly from the pixels thought of as cubical complex. Their algorithm, which applies techniques from linear algebra, demonstrates that the homology groups of a cubical space are computable. More recently, Gonzalez-Diaz, Jimenez and Medrano [5] introduced a method for computing cup products directly from a cubical approximation  $Q(I)$  of a given digital image  $I$  (no additional subdivisions are necessary).

---

*Date:* July 9, 2012.

*Key words and phrases.* Cohomology, cup product, diagonal approximation, digital image, cell complex, polygon.

Motivated by problems in high-dimensional data analysis, Kaczynski and Mrozek [10] give an algorithm for computing the cohomology ring of a cubical complex of arbitrary dimension. The implementation of the method presented in their paper is currently in progress. For a geometrical interpretation of cohomology in the context of digital images, we refer the reader to [4].

A problem that frequently arises in 3D image processing is to efficiently encode the boundary surface of a given digital object as a set of voxels. The most popular approach to this problem uses a triangulation. While triangles are combinatorially simple, and visualization of triangulated surfaces is supported by existing hardware and software, the number of triangles required is often large and the computational analysis is correspondingly slow. It is desirable, therefore, to consider combinatorial approximations with simpler combinatorics and a subsequent improvement in computational efficiency.

Given a triangulation, we introduce a technique for merging adjacent triangles to form polygonal faces of a combinatorially simpler polyhedral approximation. Using polyhedra to approximate 3D objects is well-studied problem in the field of Computational Geometry (see [1, 2, 3], for example).

In [11], Kravatz constructed a combinatorial diagonal approximation on general polygons and used it to compute cup products on closed compact orientable surfaces thought of as identification spaces of  $2n$ -gons. However, applications of Kravatz's diagonal in more general settings are limited by its dependence on a particular indexing of the vertices. Here we introduce a more general and computationally effective diagonal approximation, which does not depend upon the indexing.

The computational method introduced here can be effectively applied to any polyhedral approximation of a 3D object. Our method maximizes computational efficiency by minimizing the number of cells in a given polyhedral approximation. A Matlab implementation of the method is available on the Internet for testing<sup>1</sup>.

The paper is organized as follows: A review of the standard definitions from Algebraic Topology appears in Section 2 and the notions of cup product and diagonal approximation are reviewed in Section 3. Our main result, which appears in Section 4 as Theorem 4.1, gives an explicit formula for computing cup products on the polygons in a given polyhedral approximation  $P$  of an object embedded in 3-space. This is sufficient for computing the cohomology algebra  $H^*(P)$  since  $H^3(P) \equiv 0$ . Let  $I = (\mathbb{Z}^3, 26, 6, B)$  be a 3D digital image, let  $Q(I)$  be the associated cubical complex and let  $\partial Q(I)$  be the subcomplex of  $Q(I)$  whose maximal cells are all the quadrangles of  $Q(I)$  shared by a voxel of  $B$  (the foreground) and a voxel of  $\mathbb{Z}^3 \setminus B$  (the background). In Section 5 we show how to simplify the combinatorial structure of  $\partial Q(I)$  to obtain a polyhedral complex  $P(I)$ , which is homeomorphic to  $Q(I)$  but with fewer cells. In Section 6 we produce an algorithm for computing cup products on  $H^*(P(I))$  directly from the given combinatorics and analyze its computational complexity. The algorithm presented here is adapted from the algorithm for computing cup products in [7]. Conclusions and some ideas for future work are presented in Section 7.

## 2. STANDARD DEFINITIONS FROM ALGEBRAIC TOPOLOGY

We begin with a review of some standard definitions from Algebraic Topology (for details see [13]).

---

<sup>1</sup><http://grupo.us.es/cimagroup/imagesequences2cupproduct.zip>

Let  $S = \{S_q\}_q$  be a graded set. A  $q$ -chain of  $S$  is an element of  $S_q$ ; finite formal sums of  $q$ -chains define additive abelian group structures on  $S_q$  and  $S$ . The group of  $q$ -chains is denoted by  $C_q(S)$ ; the *chain group of  $S$*  is the graded group  $C_*(S) = \{C_q(S)\}_q$ . A *chain complex of  $S$*  consists of a chain group  $C_*(S)$  together with a square zero homomorphism  $\partial = \{\partial_q : C_q(S) \rightarrow C_{q-1}(S)\}_q$ , called the *boundary operator*. For example, consider a triangle  $\langle v_i, v_j, v_k \rangle$  with vertices  $v_i < v_j < v_k$ . The boundary of the triangle is the formal sum of its edges, that is,  $\partial_2 \langle v_i, v_j, v_k \rangle = \langle v_i, v_j \rangle + \langle v_j, v_k \rangle + \langle v_i, v_k \rangle$ .

Intuitively, a *cellular decomposition* of a 3D space  $X$  embedded in  $\mathbb{R}^3$  is a representation of  $X$  as a finite union of vertices (0-cells), edges (1-cells), polygons (2-cells), and polyhedra (3-cells), which have been glued together in such a way that the non-empty intersection of two cells is a cell. A  $k$ -cell is also referred to as a  $k$ -face. In this paper, the term *cellular complex* refers to a 3D space  $X$  embedded in  $\mathbb{R}^3$  together with a cellular decomposition. For the precise definition of a cellular complex, which is more subtle than one might expect, see [8]. An  $i$ -cell  $\sigma' \in X$  is a *facet* of a cell  $\sigma \in X$  if  $\sigma$  is an  $(i+1)$ -cell and  $\sigma'$  is a face of  $\sigma$ . A *maximal* cell of  $X$  is not a facet of any cell of  $X$ . A *cubical complex*  $Q$  is a cellular complex whose 2-cells are squares (or quadrangles) and whose 3-cells are cubes. Note that if a cube is in  $Q$ , its bounding quadrangles are in  $Q$ ; if a quadrangle is in  $Q$ , its bounding edges are in  $Q$ ; and if an edge is in  $Q$ , its endpoints are in  $Q$ .

The  $q$ -dimensional cells of a cell complex  $X$  provide an important example of a graded set. The *cellular chains of  $X$* , denoted by  $C_*(X)$ , is the chain group generated by the cells of  $X$ . The *cellular chain complex of  $X$*  is the chain complex  $(C_*(X), \partial)$ , where  $\partial$  is the linear extension of the cellular boundary. A  $q$ -chain  $\sigma \in C_q(S)$  is a  $q$ -cycle if  $\partial_q(\sigma) = 0$ ; it is a  $q$ -boundary if there is a  $(q+1)$ -chain  $\mu$  such that  $\partial_{q+1}(\mu) = \sigma$ . Referring to the triangle  $\langle v_i, v_j, v_k \rangle$  above, the 1-chain  $\sigma = \langle v_i, v_j \rangle + \langle v_j, v_k \rangle + \langle v_i, v_k \rangle$  is both a 1-cycle and a 1-boundary since  $\partial_1(\sigma) = 0$  and  $\sigma = \partial_2 \langle v_i, v_j, v_k \rangle$ . Two  $q$ -cycles  $a$  and  $a'$  are *homologous* if  $a+a'$  is a  $q$ -boundary. Let  $Z_q(S)$  and  $B_q(S)$  denote the  $q$ -cycles and  $q$ -boundaries of  $S$ , respectively. Then  $B_q(S) \subseteq Z_q(S)$  since  $\partial^2 = 0$ . The quotient  $H_q(S) = Z_q(S)/B_q(S)$  is the  $q^{\text{th}}$  *homology group of  $S$*  and the graded group  $H_*(S) = \{H_q(S)\}_q$  is the *homology of  $S$* . An element of  $H_q(S)$  is a class  $[a] := a + B_q(S)$ . The cycle  $a$  is a *representative cycle* of the class  $[a]$ . The rank of  $H_q(S)$  is called the  $i^{\text{th}}$  Betti number and is denoted by  $b_i$ .

Let  $(C_*(S), \partial)$  and  $(C_*(S'), \partial')$  be chain complexes. A homomorphism  $f = \{f_q : C_q(S) \rightarrow C_q(S')\}_q$  is a *chain map* if  $f_{q-1}\partial_q = \partial'_q f_q$  for all  $q$ . Let  $f, g : C_*(S) \rightarrow C_*(S')$  be chain maps. A *chain homotopy from  $f$  to  $g$*  is a homomorphism  $\phi = \{\phi_q : C_q(S) \rightarrow C_{q+1}(S')\}_q$  such that  $\phi_{q-1}\partial_q + \partial'_{q+1}\phi_q = f_q + g_q$  for all  $q$ . A *chain contraction of  $(C_*(S), \partial)$  to  $(C_*(S'), \partial')$*  is a tuple  $(f, g, \phi, (S, \partial), (S', \partial'))$  consisting of chain maps  $f : C_*(S) \rightarrow C_*(S')$  and  $g : C_*(S') \rightarrow C_*(S)$  such that  $fg = \text{Id}_{C_*(S')}$ , and a chain homotopy  $\phi : C_*(S) \rightarrow C_*(S)$  from  $\text{Id}_{C_*(S)}$  to  $gf$ .

**Definition 2.1.** Let  $X$  be a cell complex and let  $e$  be a  $p$ -cell of  $X$  contained in the boundary of exactly two  $(p+1)$ -cells  $E$  and  $B$ . Let  $X'$  be the cell complex obtained from  $X$  by replacing  $E$  and  $B$  with the single cell  $B' = (E \cup B) \setminus \text{int}(e)$ , and define

$$\partial'(x) = \begin{cases} \partial(x), & x \neq B' \\ \partial(E+B), & x = B'. \end{cases}$$

Then  $E$  and  $B$  have merged into  $B'$  along  $e$ .

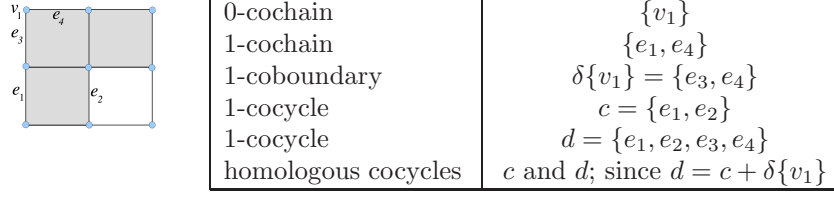


FIGURE 1. Example of cochain, cocycle and coboundary.

**Proposition 2.2.** *If cell complexes  $X$  and  $X'$  are related as in Definition 2.1, then  $H_*(X) \approx H_*(X')$ .*

*Proof.* There is a chain contraction  $(f, g, \phi, (C_*(X), \partial), (C_*(X'), \partial'))$  given by

- $f(e) = e + \partial(E)$ ,  $f(B) = B'$ ,  $f(E) = 0$ ,  $f(x) = x$  for  $x \neq e, E, B$ ;
- $g(B') = E + B$ ,  $g(x) = x$  for  $x \neq B'$ ; and
- $\phi(e) = E$ ,  $\phi(x) = 0$  for  $x \neq e$ .

Since  $\phi$  is a homotopy equivalence, the conclusion follows.  $\square$

Cochain groups are linear duals of chain groups. Given a chain complex  $(C_*(S), \partial)$ , a  $q$ -cochain is an element of  $C^q(S) := \text{Hom}(C_q(S), \mathbb{Z}_2)$ . If we index the  $q$ -cells in a cellular complex  $S$  from 1 to  $n_q$ , their corresponding duals generate  $C^q(S)$ . Thus a  $q$ -cochain is a  $\mathbb{Z}_2$ -linear combination of these duals, and can be thought of as a bit string of length  $n_q$ . The *cochain group* of  $S$  is the graded group  $C^*(S) = \{C^q(S)\}_q$ . The *coboundary operator* is the homomorphism  $\delta = \{\delta^q : C^q(S) \rightarrow C^{q+1}(S)\}_q$  defined on a  $q$ -cochain  $c$  by  $\delta^q(c) = c\partial_{q+1}$ ; then  $\delta^2 = 0$  and the *cochain complex associated with*  $(C_*(S), \partial)$  is the pair  $(C^*(S), \delta)$ .

A  $q$ -cochain  $c$  is a  $q$ -cocycle if  $\delta^q(c) = 0$ . A  $q$ -cochain  $b$  is a  $q$ -coboundary if there exists a  $(q-1)$ -cochain  $c$  such that  $b = \delta^{q-1}(c)$ . Two  $q$ -cocycles  $c$  and  $c'$  are *cohomologous* if  $c + c'$  is a  $q$ -coboundary. The  $q$ -cocycles and  $q$ -coboundaries of  $S$  are denoted by  $Z^q(S)$  and  $B^q(S)$ , respectively. Then  $B^q(S) \subseteq Z^q(S)$  since  $\delta^2 = 0$ . The quotient  $H^q(S) = Z^q(S)/B^q(S)$  is the  $q^{\text{th}}$  cohomology group of  $S$  and the graded group  $H^*(S) = \{H^q(S)\}_q$  is the cohomology of  $S$ . An element of  $H^q(S)$  is a class  $[a] := a + B^q(S)$ .

*Remark 2.3.* Since the ground ring  $\mathbb{Z}_2$  is a field, the homology and cohomology of a graded set  $S$  are isomorphic and torsion free (see [8]). Furthermore, given a chain contraction  $(f, g, \phi, (S, \partial), (S', \partial'))$ , there is a map  $\partial_\sigma f : C_*(S') \rightarrow C^*(S)$  that sends a chain  $\sigma$  to the cochain  $\partial_\sigma f$  defined by

$$\partial_\sigma f(\mu) = \begin{cases} 1, & \text{if } \sigma \text{ is a non-zero summand of } f(\mu); \\ 0, & \text{otherwise.} \end{cases}$$

Thus if  $\sigma \neq 0$ , the cochain  $\partial_\sigma f$  is supported on the subspace generated by all  $\mu \in C_*(S)$  such that  $\sigma$  is a summand of  $f(\mu)$ .

An *AT-model* for a chain complex  $(C_*(S), \partial)$  consists of a chain complex  $(C_*(H), \partial' \equiv 0)$  together with a chain contraction  $(f, g, \phi, (C_*(S), \partial), (C_*(H), \partial'))$ . An AT-model for  $(C_*(S), \partial)$  always exists and can be computed in  $\mathcal{O}(m^3)$ , where

$m = \#S$  (see [7] for details). A particular AT-model for  $(C_*(S), \partial)$  is denoted by  $(f, g, \phi, (S, \partial), H)$ .

**Proposition 2.4.** *Let  $(f, g, \phi, (S, \partial), H)$  be an AT-model for  $(C_*(S), \partial)$ .*

- i. *If  $\sigma \in H_q$ , then  $g_q(\sigma)$  represents a class in  $H_q(C_*(S))$ .*
- ii. *The map  $H_q \rightarrow H_q(S)$  given by  $\sigma \mapsto [g(\sigma)]$  extends to an isomorphism  $C_q(H) \approx H_q(S)$ .*
- iii. *The map  $H_q \rightarrow H^q(S)$  given by  $\sigma \mapsto [\partial_\sigma f]$  extends to an isomorphism  $\partial_- f : C_q(H) \approx H^q(S)$ .*

**Proposition 2.5.** *Let  $(f, g, \phi, (S, \partial), (S', \partial'))$  be a chain contraction.*

- i. *If  $(f_1, g_1, \phi_1, (S, \partial), H)$  is an AT-model for  $(C_*(S), \partial)$ , then  $(f_1 g, f g_1, f \phi_1 g, (S', \partial'), H)$  is an AT-model for  $(C_*(S'), \partial')$ .*
- ii. *If  $(f_2, g_2, \phi_2, (S', \partial'), H')$  is an AT-model for  $(C_*(S'), \partial')$ , then  $(f_2 f, g g_2, \phi + g \phi_2 f, (S, \partial), H')$  is an AT-model for  $(C_*(S), \partial)$ .*

*Proof.* (i) Observe that  $f_1 \partial(\mu) = 0$  and  $\partial g_1(\sigma) = 0$  for all  $\mu \in C_*(S)$  and  $\sigma \in C_*(H)$ . Hence  $f_1 g f g_1 = f_1(\phi \partial + \partial \phi + \text{Id}_{C_*(S)})g_1 = f_1 g_1 = \text{Id}_{C_*(H)}$  and  $f \phi_1 g \partial + \partial f \phi_1 g = f(\phi_1 \partial + \partial \phi_1)g = f(\text{Id}_{C_*(S)} + g_1 f_1)g = \text{Id}_{C_*(S')} + f g_1 f_1 g$ .

(ii) Similarly, observe that  $f_2 \partial(\mu) = 0$  and  $\partial g_2(\sigma) = 0$  for all  $\mu \in C_*(S')$  and  $\sigma \in C_*(H')$ . Hence  $f_2 f g g_2 = f_2(\text{Id}_{C_*(S')} )g_2 = f_2 g_2 = \text{Id}_{C_*(H')}$  and  $(\phi + g \phi_2 f) \partial + \partial(\phi + g \phi_2 f) = \phi \partial + \partial \phi + g(\phi_2 \partial + \partial \phi_2) f = \phi \partial + \partial \phi + g(\text{Id}_{C_*(S')} + g_2 f_2) f = \text{Id}_{C_*(S)} + g g_2 f_2 f$ .  $\square$

### 3. CUP PRODUCTS AND DIAGONAL APPROXIMATIONS

In this section we review the definitions and related notions we need to compute the cup product on the cellular cohomology  $H^*(X)$  of a cell complex  $X$ .

**Definition 3.1.** The **geometric diagonal** on  $X$  is the map  $\Delta : X \rightarrow X \times X$  defined by  $\Delta(x) = (x, x)$ . A **diagonal approximation** on  $X$  is a cellular map  $\Delta_X : X \rightarrow X \times X$  such that

- i.  $\Delta_X$  is homotopic to  $\Delta$ .
- ii. if  $e$  is a cell of  $X$ , then  $\Delta_X(e) \subseteq e \times e$ .
- iii.  $\Delta_X$  extends to a chain map  $C_*(X) \rightarrow C_*(X) \otimes C_*(X)$ , in which case the geometric boundary  $\partial$  extends to a coderivation of  $\Delta_X$ , i.e.,

$$\Delta_X \partial = (\text{Id} \otimes \partial + \partial \otimes \text{Id}) \Delta_X.$$

A diagonal approximation  $\Delta_X$  always exists by the Cellular Approximation Theorem (see [8]). If  $\Delta_X$  is coassociative,  $(C_*(X), \partial, \Delta_X)$  is a differential graded coalgebra (DGC).

**Example 3.2.** The *Alexander-Whitney (A-W) diagonal approximation*  $\Delta_s$  on the  $n$ -simplex  $s_n = \langle 0, 1, \dots, n \rangle$ ,  $n \geq 0$ , is defined by

$$\Delta_s \langle 0, 1, \dots, n \rangle = \sum_{i=0}^n \langle 0, \dots, i \rangle \otimes \langle i, \dots, n \rangle$$

and extends to a strictly coassociative, homotopy cocommutative chain map  $\Delta_s : C_*(s_n) \rightarrow C_*(s_n) \otimes C_*(s_n)$ . Thus  $(C_*(s_n), \partial, \Delta_s)$  is a DGC for each  $n$  (see [8]).

The 0-*cube* is the single point 0, the 1-*cube* is the unit interval  $I = [0, 1]$ , and the  $n$ -*cube* is the  $n$ -fold Cartesian product  $I^n = I \times \cdots \times I$ . A cell of  $I^n$  is an  $n$ -fold Cartesian product  $u_1 \cdots u_n \in \{0, 1, I\}^{\times n}$ .

**Example 3.3.** The *Serre diagonal approximation*  $\Delta_I$  on the  $n$ -*cube*  $I^n$  is given by

$$\Delta_I(0) = 0 \otimes 0 \quad \text{and} \quad \Delta_I(I^n) = (0 \otimes I + I \otimes 1)^{\times n} \quad \text{for } n \geq 1,$$

where  $a \otimes b \times c \otimes d = ac \otimes bd$ . In particular,

$$\Delta_I(I^2) = (0 \otimes I + I \otimes 1) \times (0 \otimes I + I \otimes 1) = 00 \otimes II + 0I \otimes I1 + I0 \otimes 1I + II \otimes 11.$$

Let  $0' = I$  and  $I' = 1$ ; then for  $n \geq 1$  we obtain

$$\Delta_I(I^n) = \sum_{u_1 \cdots u_n \in \{0, I\}^{\times n}} u_1 \cdots u_n \otimes u'_1 \cdots u'_n.$$

Again,  $\Delta_I$  extends to a strictly coassociative, homotopy cocommutative chain map  $\Delta_I : C_*(I^n) \rightarrow C_*(I^n) \otimes C_*(I^n)$  and  $(C_*(I^n), \partial, \Delta_I)$  is a DGC for each  $n$  (see [15]).

Let  $\Delta_X : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$  be a diagonal approximation and let  $(f, g, \phi, (X, \partial), H)$  be an AT-model for  $(C_*(X), \partial)$ . Then  $C_*(H) = H_*(X) \approx H^*(X)$  by Remark 2.3 and Proposition 2.4. Given classes  $a \in H^p(X)$  and  $b \in H^q(X)$ , there exist unique chains  $\alpha = (\partial_- f)^{-1}(a) \in C_p(H)$  and  $\beta = (\partial_- f)^{-1}(b) \in C_q(H)$  such that  $a = [\partial_\alpha f]$  and  $b = [\partial_\beta f]$ . The *cup product* of representative cocycles  $\partial_\alpha f$  and  $\partial_\beta f$  is the  $(p+q)$ -cocycle defined on  $\gamma \in C_{p+q}(X)$  by

$$(\partial_\alpha f \smile_X \partial_\beta f)(\gamma) = \cdot (\partial_\alpha f \otimes \partial_\beta f) \Delta_X(\gamma),$$

where “ $\cdot$ ” denotes multiplication in  $\mathbb{Z}_2$ , and the *cup product* of classes  $a = [\partial_\alpha f] \in H^p(X)$  and  $b = [\partial_\beta f] \in H^q(X)$  is the class

$$a \smile_X b = [\partial_\alpha f \smile_X \partial_\beta f] \in H^{p+q}(X).$$

The cup product on  $H^*(X)$  is a bilinear operation with unit. If  $\Delta_X$  is homotopy coassociative,  $(H^*(X), \smile_X)$  is a graded algebra with unity; if  $\Delta_X$  is homotopy cocommutative,  $(H^*(X), \smile_X)$  is (graded) commutative. In particular,  $\smile_s$  and  $\smile_I$  are commutative products with units.

**Definition 3.4.** Let  $X$  and  $X'$  be cell complexes, let  $\Delta_X : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$  be a diagonal approximation, let  $(f, g, \phi, (X, \partial), (X', \partial'))$  be a chain contraction, and let  $(f', g', \phi', (X', \partial'), H')$  be an AT-model for  $C_*(X')$ . The **induced coproduct** on  $X'$  is the chain map

$$\tilde{\Delta}_{X'} := (f \otimes f)(\Delta_X)g : C_*(X') \rightarrow C_*(X') \otimes C_*(X').$$

**Proposition 3.5.** *Under the conditions of Definition 3.4, the induced coproduct  $\tilde{\Delta}_{X'}$  is homotopy cocommutative whenever  $\Delta_X$  is homotopy cocommutative.*

*Proof.* Let  $\tau : A \otimes A \rightarrow A \otimes A$  denote the twisting isomorphism given by  $\tau(a \otimes b) = b \otimes a$ , and choose a chain homotopy  $h : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$  such that

$$\tau \Delta_X + \Delta_X = (\partial \otimes \text{Id} + \text{Id} \otimes \partial) h + h \partial.$$

Then  $(f \otimes f)hg$  is a chain homotopy from  $\tau\tilde{\Delta}_{X'}$  to  $\tilde{\Delta}_{X'}$  as the following calculation demonstrates:

$$\begin{aligned}\tau\tilde{\Delta}_{X'} + \tilde{\Delta}_{X'} &= \tau(f \otimes f)\Delta_X g + (f \otimes f)\Delta_X g \\ &= (f \otimes f)(\tau\Delta_X + \Delta_X)g \\ &= (f \otimes f)((\partial \otimes \text{Id} + \text{Id} \otimes \partial)h + h\partial)g \\ &= (\partial \otimes \text{Id} + \text{Id} \otimes \partial)(f \otimes f)hg + (f \otimes f)hg\partial.\end{aligned}$$

□

In practice, we compute “cup products” in homology via an AT-model  $(f, g, \phi, (X, \partial), H)$  for  $C_*(X)$ . Having obtained a basis for  $C_*(H) \approx H^*(X)$ , we define the homology cup product as follows:

**Definition 3.6.** Let  $X$  be a cell complex, let  $\Delta_X : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$  be a diagonal approximation, and let  $(f, g, \phi, (X, \partial), H)$  an AT-model for  $C_*(X)$ . Given  $\alpha \in H_p$  and  $\beta \in H_q$ , define

$$\alpha \smile_X \beta := \sum_{\gamma \in H_{p+q}} \left\{ \cdot (\partial_\alpha f \otimes \partial_\beta f)(\tilde{\Delta}_X)g(\gamma) \right\} \gamma \in C_*(H_{p+q}).$$

**Proposition 3.7.** Let  $X$  and  $X'$  be cell complexes, let  $\Delta_X : C_*(X) \rightarrow C_*(X) \otimes C_*(X)$  be a diagonal approximation, let  $(f, g, \phi, (X, \partial), (X', \partial'))$  be a chain contraction, and let  $(f', g', \phi', (X', \partial'), H')$  be an AT-model for  $C_*(X')$ . Then for  $\alpha \in H'_p$  and  $\beta \in H'_q$  we have

$$\alpha \smile_{X'} \beta = \alpha \smile_X \beta.$$

*Proof.* Since  $(f'f, gg', \phi + g\phi'f, (X, \partial), H')$  is an AT-model for  $C_*(X)$  by Proposition 2.5, part (ii), and  $C_*(H^*(X)) \approx H^*(X')$  by Remark 2.3 we have

$$\begin{aligned}\alpha \smile_{X'} \beta &= \sum_{\gamma \in H'_{p+q}} \left\{ \cdot (\partial_\alpha f' \otimes \partial_\beta f')(\tilde{\Delta}_{X'})g'(\gamma) \right\} \gamma \\ &= \sum_{\gamma \in H'_{p+q}} \left\{ \cdot (\partial_\alpha f' \otimes \partial_\beta f')(f \otimes f)(\Delta_X)gg'(\gamma) \right\} \gamma \\ &= \sum_{\gamma \in H'_{p+q}} \left\{ \cdot (\partial_\alpha f'f \otimes \partial_\beta f'f)(\Delta_X)gg'(\gamma) \right\} \gamma = \alpha \smile_X \beta.\end{aligned}$$

□

#### 4. CUP PRODUCTS ON 3D POLYHEDRAL APPROXIMATIONS

Traditionally, one uses the standard formulas in [13, 15] to compute cup products on a simplicial or cubical complex. In this section, we show how to compute cup products on a 3-dimensional polyhedral approximation  $P$  of a cell complex embedded in  $\mathbb{R}^3$ .

Arbitrarily label the  $n$  vertices of  $P$  with the integers  $\{1, \dots, n\}$  and represent a  $k$ -gon  $p \subset P$  by an ordered  $k$ -tuple of integers  $p = \langle i_1, \dots, i_k \rangle$ , where  $i_1 = \min\{i_1, \dots, i_k\}$ ,  $i_1$  is adjacent to  $i_k$ , and  $i_j$  is adjacent to  $i_{j+1}$  for  $1 < j < k$ . The statement in Theorem 4.1 will use the following notation: The symbol  $i_{m(k)} := \max\{i_2, \dots, i_k\}$ ; the  $\mathbb{Z}_2$ -coefficient  $\lambda_j = 0$  if and only if  $i_j < i_{j+1}$ ; and edges of interest are denoted by  $\{u_j = \langle i_1, i_j \rangle\}_{2 \leq j \leq k}$  and  $\{e_j = \langle i_j, i_{j+1} \rangle\}_{2 \leq j \leq k-1}$ .



**Theorem 4.1.** *Let  $P$  be a polyhedral approximation. The A-W diagonal induces a coproduct  $\tilde{\Delta}_P$  on each polygon  $p = \langle i_1, \dots, i_k \rangle \subset P$  given by*

$$(1) \quad \begin{aligned} \tilde{\Delta}_P(p) = & \langle i_1 \rangle \otimes p + p \otimes \langle i_{m(k)} \rangle + \sum_{j=2}^{m(k)-1} (u_2 + e_2 + \dots + e_{j-1} + \lambda_j e_j) \otimes e_j \\ & + \sum_{j=m(k)}^{k-1} [(1 + \lambda_j) e_j + e_{j+1} + \dots + e_{k-1} + u_k] \otimes e_j. \end{aligned}$$

*Proof.* Consider the triangulation  $\{T_{j-1} = \langle i_1, i_j, i_{j+1} \rangle\}_{2 \leq j \leq k-1}$  of  $p$  and note that the A-W diagonal on  $T_{j-1}$  is given by

$$(2) \quad \begin{aligned} \Delta_s(T_{j-1}) = & \lambda_j (\langle i_1 \rangle \otimes T_{j-1} + T_{j-1} \otimes \langle i_j \rangle + u_{j+1} \otimes e_j) \\ & + (1 + \lambda_j) (\langle i_1 \rangle \otimes T_{j-1} + T_{j-1} \otimes \langle i_{j+1} \rangle + u_j \otimes e_j). \end{aligned}$$

Our strategy is to inductively merge these triangles until  $p$  is recovered and the induced coproduct is obtained. We proceed by induction on  $j$ .

When  $j = 2$ , set  $p_1 = T_1$  and note that either  $i_2 < i_3$ , in which case  $i_{m(3)} = i_3$  and  $\lambda_2 = 0$ , or  $i_2 > i_3$ , in which case  $i_{m(3)} = i_2$  and  $\lambda_2 = 1$ . If  $i_{m(3)} = i_3$ , formula (1) gives the non-primitive terms  $[u_2 + \lambda_2 e_2] \otimes e_2$ . Since  $\lambda_2 = 0$  this expression reduces to  $u_2 \otimes e_2 = \lambda_2 u_3 \otimes e_3 + (1 + \lambda_2) u_2 \otimes e_2$ . On the other hand, if  $i_{m(3)} = i_2$ , formula (1) gives the non-primitive terms  $[(1 + \lambda_2) e_2 + u_3] \otimes e_3$ . Since  $\lambda_2 = 1$  this expression reduces to  $u_3 \otimes e_3 = \lambda_2 u_3 \otimes e_3 + (1 + \lambda_2) u_2 \otimes e_2$ . In either case, formula (1) agrees with the A-W diagonal on  $p_1$ .

Now assume that for some  $j \geq 3$ , formula (1) holds on  $p_{j-2} = \langle i_1, \dots, i_j \rangle$ . Merge  $p_{j-2}$  and  $T_{j-1}$  along  $u_j$  and obtain  $p_{j-1} = \langle i_1, \dots, i_{j+1} \rangle$ ; we claim that formula (1) also holds on  $p_{j-1}$ . Recall that

$$\tilde{\Delta}_P(p_{j-1}) = (f \otimes f) \tilde{\Delta}_s g(p_{j-1}) = (f \otimes f) \tilde{\Delta}_s(p_{j-2} + T_{j-1}).$$

Either  $i_{j+1} > i_{m(j)}$ , in which case  $\lambda_j = 0$  and  $i_{m(j+1)} = i_{j+1}$ , or  $i_{m(j)} > i_{j+1}$ , in which case  $i_{m(j+1)} = i_{m(j)}$ . First assume that  $i_{j+1} > i_{m(j)}$ . Following the proof of Proposition 3.3, define  $f(u_j) = u_2 + e_2 + \dots + e_{j-1}$ ,  $f(p_{j-2}) = 0$ , and  $f(T_{j-1}) = p_{j-1}$ . Then



$$\begin{aligned}
\tilde{\Delta}_P(p_{j-1}) &= (f \otimes f) \{ \langle i_1 \rangle \otimes T_{j-1} + T_{j-1} \otimes \langle i_{j+1} \rangle + u_j \otimes e_j \\
&\quad + \langle i_1 \rangle \otimes p_{j-2} + p_{j-2} \otimes \langle i_{m(j)} \rangle + \sum_{s=2}^{m(j)-1} (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s \\
&\quad + \sum_{s=m(j)}^{j-1} [(1 + \lambda_s) e_s + e_{s+1} + \cdots + e_{j-1} + u_j] \otimes e_s \} \\
&= \langle i_1 \rangle \otimes p_{j-1} + p_{j-1} \otimes \langle i_{j+1} \rangle \\
&\quad + (u_2 + e_2 + \cdots + e_{j-1}) \otimes e_j + \sum_{s=2}^{m(j)-1} (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s \\
&\quad + \sum_{s=m(j)}^{j-1} [(1 + \lambda_s) e_s + e_{s+1} + \cdots + e_{j-1} + (u_2 + e_2 + \cdots + e_{j-1})] \otimes e_s \\
&= \langle i_1 \rangle \otimes p_{j-1} + p_{j-1} \otimes \langle i_{j+1} \rangle + \sum_{s=2}^j (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s,
\end{aligned}$$

which verifies formula (1) in this case. On the other hand, if  $i_{m(j)} > i_{j+1}$  define  $f(u_j) = e_j + u_{j+1}$ ,  $f(p_{j-2}) = p_{j-1}$ , and  $f(T_{j-1}) = 0$ . Then

$$\begin{aligned}
\tilde{\Delta}_P(p_{j-1}) &= (f \otimes f) \{ \lambda_j (\langle i_1 \rangle \otimes T_{j-1} + T_{j-1} \otimes \langle i_j \rangle + u_{j+1} \otimes e_j) \\
&\quad + (1 + \lambda_j) (\langle i_1 \rangle \otimes T_{j-1} + T_{j-1} \otimes \langle i_{j+1} \rangle + u_j \otimes e_j) \\
&\quad + \langle i_1 \rangle \otimes p_{j-2} + p_{j-2} \otimes \langle i_{m(j)} \rangle + \sum_{s=2}^{m(j)-1} (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s \\
&\quad + \sum_{s=m(j)}^{j-1} [(1 + \lambda_s) e_s + e_{s+1} + \cdots + e_{j-1} + u_j] \otimes e_s \} \\
&= \langle i_1 \rangle \otimes p_{j-1} + p_{j-1} \otimes \langle i_{m(j)} \rangle + \lambda_j u_{j+1} \otimes e_j \\
&\quad + (1 + \lambda_j) (e_j + u_{j+1}) \otimes e_j + \sum_{s=2}^{m(j)-1} (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s \\
&\quad + \sum_{s=m(j)}^{j-1} [(1 + \lambda_s) e_s + e_{s+1} + \cdots + e_{j-1} + (e_j + u_{j+1})] \otimes e_s \\
&= \langle i_1 \rangle \otimes p_{j-1} + p_{j-1} \otimes \langle i_{m(j)} \rangle + \sum_{s=2}^{m(j)-1} (u_2 + e_2 + \cdots + \lambda_s e_s) \otimes e_s \\
&\quad + \sum_{s=m(j)}^j [(1 + \lambda_s) e_s + e_{s+1} + \cdots + e_j + u_{j+1}] \otimes e_s,
\end{aligned}$$

which verifies formula (1) in this case as well and completes the proof.  $\square$

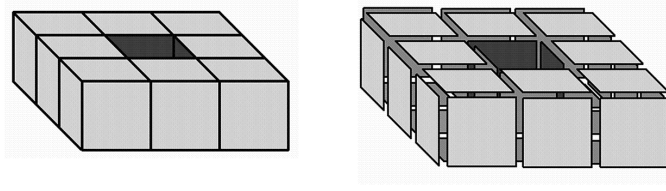


FIGURE 2. Left: A digital image  $I = (\mathbb{Z}^3, 26, 6, B)$ ; the set  $B$  consists of 8 unit cubes (voxels). Right: The quadrangles of  $\partial Q(I)$ .

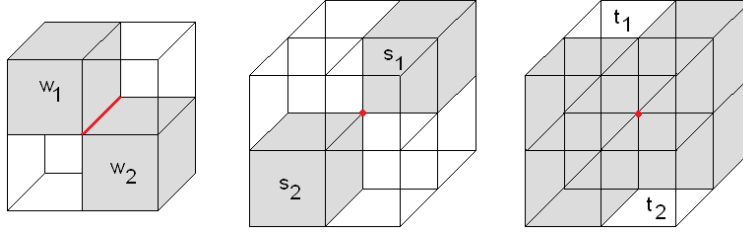


FIGURE 3. Critical configurations (i), (ii) and (iii) (modulo reflections and rotations).

### 5. 3D DIGITAL PICTURES AND CELLULAR COMPLEXES

Let  $I$  be a 3D digital image and let  $Q(I)$  be an associated cubical complex. In this section we introduce a simplification procedure, which produces a cellular complex  $P(I)$  homeomorphic to  $\partial Q(I)$  with significantly fewer cells.

Consider a 3D binary digital picture  $I = (\mathbb{Z}^3, 26, 6, B)$ , where  $\mathbb{Z}^3$  is the underlying grid and  $B$  (the foreground) is a finite set of points of the grid fixing the 26-adjacency for the points of  $B$  and the 6-adjacency for the points of  $\mathbb{Z}^3 \setminus B$  (the background). The cells of  $Q(I)$  are unit cubes centered at the points of  $B$  with faces parallel to the coordinate planes (called the *voxels* of  $I$ ), together with their quadrangles, edges, and vertices. The subcomplex  $\partial Q(I)$  consists of all cells of  $Q(I)$  that are facets of exactly one (maximal) cell of  $Q(I)$ , and their faces. Note that the maximal cells of  $\partial Q(I)$  are all the quadrangles of  $Q(I)$  shared by a voxel of  $B$  and a voxel of  $\mathbb{Z}^3 \setminus B$  (see Figure 2).

We perform a simplification process in  $\partial Q(I)$  to produce a cellular complex  $P(I)$  homeomorphic to  $\partial Q(I)$  such that the maximal cells of  $P(I)$  are polygons. But first, we need a definition.

**Definition 5.1.** A vertex  $v \in \partial Q(I)$  is **critical** if any of the following conditions is satisfied:

- i.  $v$  is a face of some edge  $e$  shared by four cubes, exactly two of which intersect along  $e$  and lie in  $Q(I)$  (see cubes  $w_1$  and  $w_2$  in Figure 3).
- ii.  $v$  is shared by eight cubes, exactly two of which are corner-adjacent and contained in  $Q(I)$  (see cubes  $s_1$  and  $s_2$  in Figure 3).

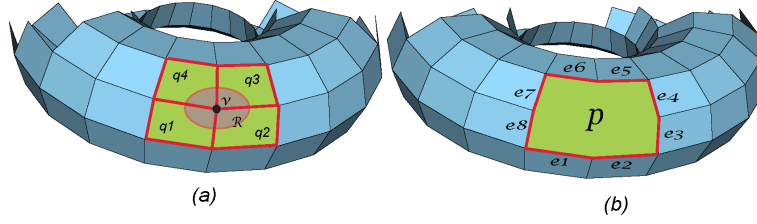


FIGURE 4. (a)  $N_v \leftarrow \{q_1, q_2, q_3, q_4\}$ ; (b) facets of  $p \leftarrow \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ .

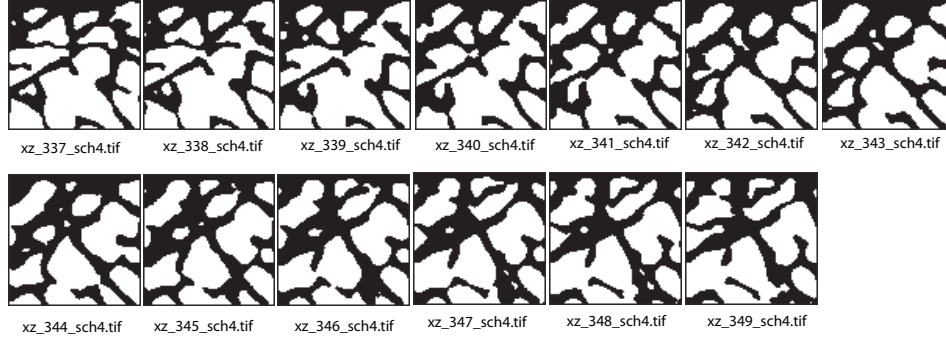


FIGURE 5. A  $\mu$ MRI of a trabecular bone.

- iii.  $v$  is shared by eight cubes, exactly two of which are corner-adjacent and not contained in  $Q(I)$  (cubes  $t_1$  and  $t_2$  in Figure 3).

A non-critical vertex of  $\partial Q(I)$  lies in a neighborhood of  $\partial Q(I)$  homeomorphic to  $\mathbb{R}^2$  (see [12]).

Algorithm 5.2 presented below processes the non-critical vertices of  $\partial Q(I)$  to obtain the cellular complex  $P(I)$ . Initially,  $P(I) = \partial Q(I)$ . For a vertex  $v \in P(I)$ , let  $N_v$  be the set of 2-cells  $q \in P(I)$  incident to the vertex  $v$ . If  $N_v$  defines a region  $R_v$  homeomorphic to a disc, then  $N_v$  is replaced by a new 2-cell  $p$  in  $P(I)$ , which is the union of the cells of  $N_v$ . The edges of  $P(I)$  incident to  $v$  and the vertex  $v$  are removed from  $P(I)$  (see Figure 4). Observe that the maximal cells of the final cellular complex  $P(I)$  are polygons and  $P(I)$  has fewer cells than  $\partial Q(I)$ . We need some terminating conditions. For example: (1) terminate when the number of edges of the polygons in  $P(I)$  reach some specified maximum; or (2) terminate after merging the set  $N_v$  of coplanar 2-cells of  $P(I)$ . Example 5.3 demonstrates the differences that arise from these different terminating conditions.

**Algorithm 5.2.** *Algorithm to obtain the cellular complex  $P(I)$ .*

INPUT: the cubical complex  $Q(I)$ .

Initially,  $P(I) := \partial Q(I)$ ;

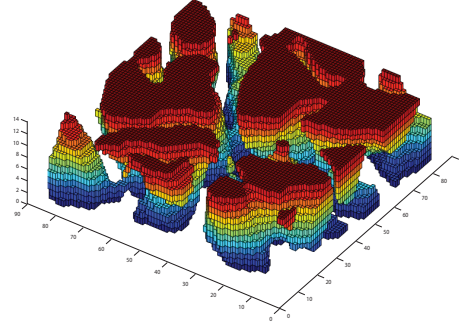
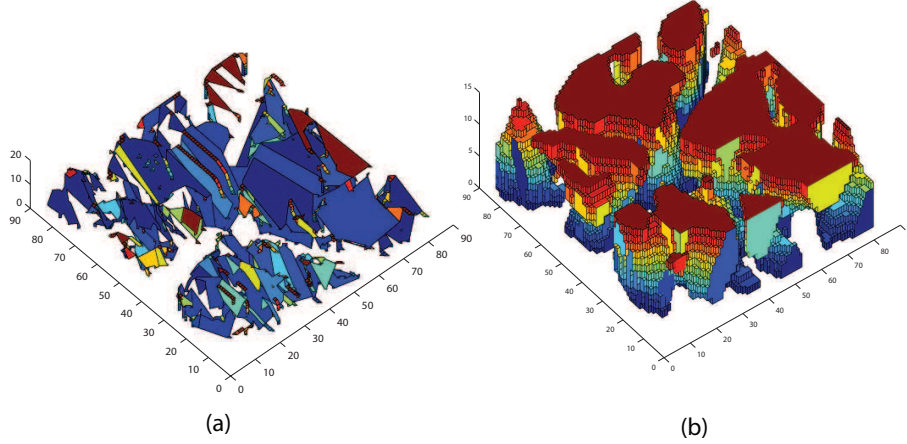
$V :=$  ordered set of non-critical vertices of  $\partial Q(I)$ .

While  $\exists v \in V$  such that  $R_v$  is homeomorphic to a disc do

    remove  $v$  from  $P(I)$  and  $V$ ;

    remove the edges incident to  $v$  from  $P(I)$ ;

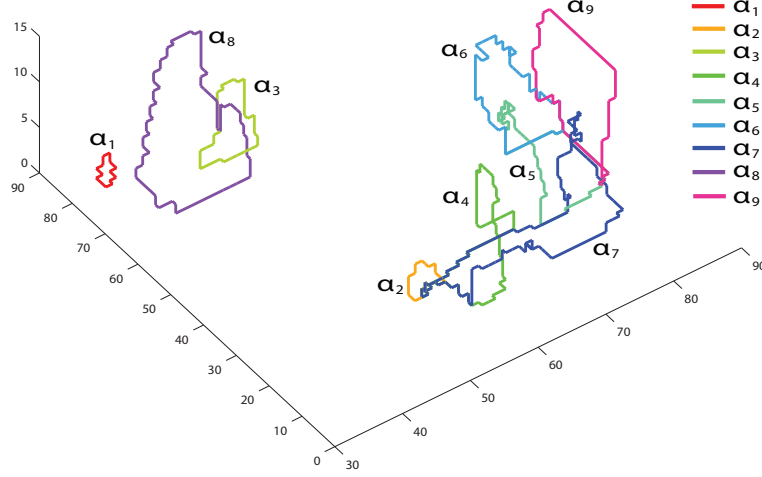
    remove the 2-cells of  $N_v$  from  $P(I)$ ;

FIGURE 6. The cubical complex  $\partial Q(\mu I)$ .FIGURE 7. (a) The cellular complex  $P(\mu I)$  for 10 edges as upper bound on  $p \in P(\mu I)$ . (b) The cellular complex  $P(\mu I)$  preserving geometry.

add a new 2-cell  $p$  to  $P(I)$   
     which is the union of the 2-cells of  $N_v$ .  
 OUTPUT: The cellular complex  $P(I)$ .

Observe that Algorithm 5.2 uses the ordering on the set of non-critical vertices  $V \subset \partial Q(I)$  to select the next non-critical vertex. To the best of our knowledge, this is the first algorithm to appear that produces a cellular complex with polygonal maximal cells by removing non-critical vertices.

**Example 5.3.** Let  $\mu I$  be a  $\mu$ MRI of a trabecular bone of size:  $85 \times 85 \times 13$  voxels (see Figure 5 in which  $\mu I$  is given by a sequence of 13 2D digital images of size  $85 \times 85$ ). The number of quadrangles in  $\partial Q(\mu I)$  is 23136 (see Figure 6). After applying Algorithm 5.2 to  $\partial Q(\mu I)$  with the terminating condition 1 (the number of the edges of the polygons in  $P(\mu I)$  is at most 10), the number of polygons of  $P(\mu I)$  is 1044 (see Figure 7.a). If we only consider the set of coplanar polygons  $N_v$ , then the number of polygons of  $P(\mu I)$  after applying Algorithm 5.2 is 11895 (see Figure 7.b).

FIGURE 8. Representative 1-cycles on the cellular complex  $P(\mu I)$ .

Now, consider the output  $P(I)$  of Algorithm 5.2. The chain complex  $(C_*(P(I)), \partial)$  can be described as follows:

- The chain group  $C_q(P(I))$  is generated by the  $q$ -cells of  $P(I)$ .
- The value of  $\partial_q : C_q(P(I)) \rightarrow C_{q-1}(P(I))$  on a  $q$ -cell is the sum of its facets.
- The boundary of a sum of  $q$ -cells is the sum of their boundaries.

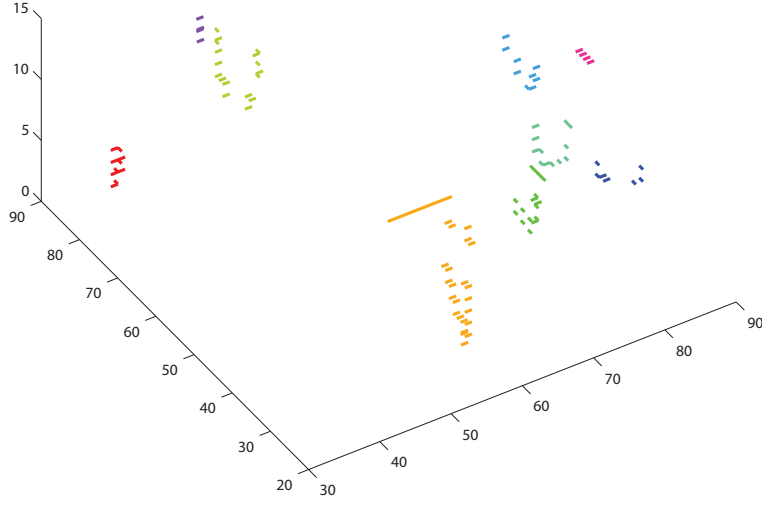
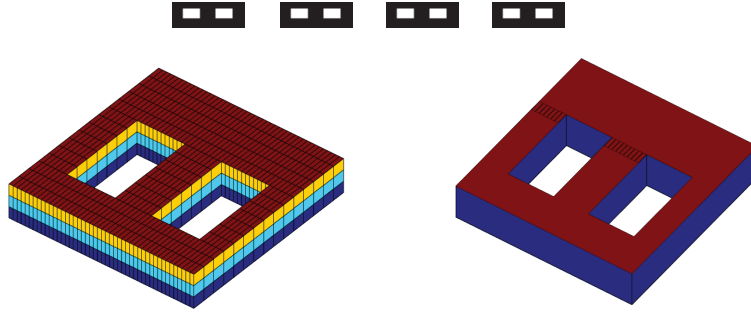
**Example 5.4.** Consider the cellular complex  $P(\mu I)$  displayed in Figure 7.b, which was produced by an application of Algorithm 5.2 to a  $\mu$ MRI of a trabecular bone of size  $85 \times 85 \times 10$  voxels. Table 1 exhibits the homology obtained by computing an AT-model for  $P(\mu I)$ . These results count the number of connected components, holes and cavities. Representative 1-cycles are shown in Figure 8.

TABLE 1. Results of the homology groups computation for the cellular complex  $P(\mu I)$  shown in Figure 7.b. (see Figure 8).

Cellular Complex	$b_0$	$b_1$	$b_2$
$P(\mu I)$	10	9	17

## 6. COMPUTING THE COHOMOLOGY ALGEBRA $H^*(P(I))$

The following algorithm computes cup products in  $H^*(P(I))$  in terms of  $\tilde{\Delta}_P$ . Since  $H^3(P(I)) \equiv 0$ , all non-trivial cup products in  $H^*(P(I))$  are products of 1-cocycles.

FIGURE 9. Representative 1-cocycles on the cellular complex  $P(\mu I)$ .FIGURE 10. Left: cell complex  $A$ . Right: cell complex  $B$ .

**Algorithm 6.1.** *Computing Cup Products on  $P(I)$ .*

INPUT: A cubical complex  $Q(I)$  associated to a 3D digital image  $I$ .

Compute a polyhedral cell complex  $P(I)$  homeomorphic to  $\partial Q(I)$   
using Algorithm 5.2.

Compute an AT-model  $(f, g, \phi, (P(I), \partial), H)$  for  $(C_*(P(I)), \partial)$   
using the algorithm given in [7].

Initially let  $A$  be a  $\frac{b_1(b_1+1)}{2} \times b_2$  zero matrix.

Let  $H_1 = \{\alpha_1, \dots, \alpha_{b_1}\}$  and  $H_2 = \{\gamma_1, \dots, \gamma_{b_2}\}$ .

For  $i = 1$  to  $b_1$  do

For  $j = i$  to  $b_1$  do

For  $k = 1$  to  $b_2$  do

$$A(i+j, k) = \cdot(\partial_{\alpha_i} f \otimes \partial_{\alpha_j} f)(\tilde{\Delta}_P)g(\gamma_k);$$

OUTPUT: the matrix  $A$ .

Observe that

$$\sum_{k=1}^{b_2} \{A(i+j, k)\} \gamma_k = \alpha_i \smile_P \alpha_j$$

by Definition 3.6 and the matrix  $A$  is symmetric since  $\tilde{\Delta}_P$  is homotopy cocommutative by Proposition 3.5.

Let  $m$  be the number of voxels of  $I$ . Since the number of cells of  $Q(I)$  is at most  $27 \cdot m$ , i.e.,  $O(m)$ , the number of critical vertices of  $\partial Q(I)$  is  $O(m)$ . The overall computational complexity of Algorithm 6.1 is  $O(m + n^3 + b_2 \cdot n \cdot k \cdot b_1^2)$ , where  $b_i$  is the  $i^{th}$  Betti number; its three components are determined as follows:

- The computational complexity of Algorithm 5.2 to obtain the cellular complex  $P(I)$  is  $O(m)$  since, in the worst case, the number of edges in  $P(I)$  incident to a non-critical vertex  $v \in P(I)$  is 6 and the number of 2-cell of  $N_v$  is 12.
- Let  $n$  be the number of cells of  $P(I)$  and  $k$  the number of vertices of the biggest polygon of  $P(I)$ . The computational complexity of the algorithm given in [7] to obtain an AT-model  $(f, g, \phi, (P(I), \partial), H)$  for  $(C_*(P(I)), \partial)$  is  $O(n^3)$ .
- The complexity to compute a row of  $A$  is at most  $O(n \cdot k^2 \cdot b_1^2)$ , since for a fixed  $\gamma \in H_2$ ,  $g(\gamma)$  has at most  $n$  summands,  $\tilde{\Delta}_P$  has  $k$  summands and for a 1-cell  $\sigma \in P(I)$ , and  $f(\sigma)$  has at most  $b_1$  summands.

Finally, observe that, in most cases,  $k \ll n \ll m$  and  $b_\ell \ll n \ll m$  for  $\ell = 1, 2$ . Then, the overall complexity in most cases is  $O(n^3)$ .

**Example 6.2.** Starting from the results obtained in Example 5.4 and applying the formula given in Theorem 4.1, representative 1-cocycles are shown in Figure 9. Table 2 shows the cup product.

TABLE 2. Results of the computation of the cup product on the cellular complex shown in Figure 7.b. Only non-null results are shown.

	$(\alpha_2, \alpha_4)$	$(\alpha_3, \alpha_8)$	$(\alpha_5, \alpha_7)$	$(\alpha_6, \alpha_9)$	$(\alpha_7, \alpha_9)$
$\gamma_{16}$	1	1	1	1	1

Table 3 illustrates the dramatic improvement in computational efficiency realized by removing faces and non-critical vertices.

**Example 6.3.** Let  $DBrain$  be a 3D digital image obtained from a MR imaging of the brain. After applying Algorithm 5.2 with terminating condition (2), i.e., terminate after merging the set  $N_v$  of coplanar 2-cells of  $\partial Q(DBrain)$ , we obtain a cellular complex  $P(DBrain)$  with 2433 polygons (see Figure 11.Left). After computing an AT-model  $(f, g, \phi, (P(DBrain), \partial), H)$  for  $(C_*(P(DBrain)), \partial)$  we obtain  $b_0 = 2$ ,  $b_1 = 19$  and  $b_2 = 2$ . Representative 1-cycles are shown in Figure 11.Right. Table 4 shows the resulting cup product.

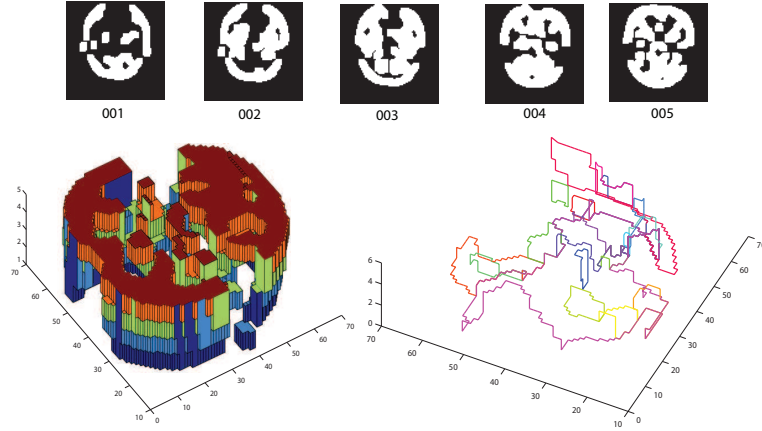


TABLE 3. Time (in seconds) and the results of the computation of the cup product on the cellular complexes shown in Figure 10.

Cell complex	Number of 2-cells			Time to compute the cup product		
$A$	1638			28.00 sec.		
$B$	46			1.04 sec.		
	$(\alpha_1, \alpha_2)$	$(\alpha_1, \alpha_3)$	$(\alpha_1, \alpha_4)$	$(\alpha_2, \alpha_3)$	$(\alpha_2, \alpha_4)$	$(\alpha_3, \alpha_4)$
$\gamma$	0	1	1	0	1	0

TABLE 4. Results of the computation of the cup product for the cellular complex shown in Figure 11. Only non-null results are shown.

	$\alpha_1\alpha_8$	$\alpha_2\alpha_3$	$\alpha_2\alpha_5$	$\alpha_2\alpha_{17}$	$\alpha_3\alpha_5$	$\alpha_3\alpha_6$	$\alpha_3\alpha_7$	$\alpha_3\alpha_9$
$\gamma_2$	1	1	1	1	1	1	1	1
	$\alpha_3\alpha_{13}$	$\alpha_5\alpha_6$	$\alpha_5\alpha_7$	$\alpha_5\alpha_9$	$\alpha_5\alpha_{13}$	$\alpha_6\alpha_{12}$	$\alpha_6\alpha_{13}$	$\alpha_6\alpha_{16}$
$\gamma_2$	1	1	1	1	1	1	1	1
	$\alpha_8\alpha_{19}$	$\alpha_9\alpha_{17}$	$\alpha_{11}\alpha_{12}$	$\alpha_{12}\alpha_{19}$	$\alpha_{13}\alpha_{15}$	$\alpha_{14}\alpha_{18}$	$\alpha_{16}\alpha_{19}$	$\alpha_{18}\alpha_{19}$
$\gamma_2$	1	1	1	1	1	1	1	1

FIGURE 11. Left: cell complex  $P(DBrain)$ . Right: Representative 1-cycles.

## 7. CONCLUSIONS AND PLANS FOR FUTURE WORK

Given a 3D digital image  $I$ , we have formulated the cup product on the cohomology of the cellular complex  $P(I)$  obtained by simplifying the cubical complex  $\partial Q(I)$ . The algorithm proposed here is valid for any encoding of a 3D digital object given as a set of polyhedra. Our ultimate goal is to compute cup products on a general  $n$ -dimensional cellular complex over a general ring directly from the given

combinatorial structure (without subdivisions). Our strategy is to apply some standard geometric constructions such as forming quotients, taking Cartesian products, and merging cells.

## REFERENCES

- [1] Argawal P.K., Suri S., Surface Approximation and Geometric Partitions. SIAM Journal on Computing 27 (4) (1998), 1016-1035.
- [2] Bronnimann H., Goodrich M.T., Almost Optimal Set Covers in Finite VC-dimension. Discrete and Computational Geometry 14 (1995), 263-279.
- [3] Das G., Goodrich M.T., On the Complexity of Optimization Problem for 3D Convex Polyhedra and Decision Trees. Computational Geometry, Theory and Applications 8 (1997), 123-137.
- [4] Gonzalez-Diaz R., Ion A., Iglesias-Ham M., Kropatsch W, Invariant Representative Cocycles of Cohomology Generators using Irregular Graph Pyramids. Computer Vision and Image Understanding 115 (7), 1011-1022
- [5] Gonzalez-Diaz R., Jimenez M.J., Medrano B., Cohomology Ring of 3D Photographs. Int. Journal of of Imaging Systems and Technology, 21 (2011), 76-85.
- [6] Gonzalez-Diaz R., Lamar J., Umble R., Cup products on polyhedral approximations of 3D digital images. In Proc. of the 14th int. conf. on Combinatorial Image Analysis (IWCIA'11), LNCS 6636 (2011), 107-119
- [7] Gonzalez-Diaz R., Real P., On the Cohomology of 3D Digital Images. Discrete Applied Math. 147 (2005), 245-263
- [8] Hatcher A., Algebraic Topology. Cambridge University Press, 2002.
- [9] Kaczynski T., Mischaikow K., Mrozek M., Computational Homology. Applied Mathematical Sciences v. 157 (2004).
- [10] Kaczynski T., Cubical Cohomology ring, An Algorithmic Approach. Preprint. <http://pages.usherbrooke.ca/tkaczynski/preprints/cubcohomTKMM-preprint.pdf>
- [11] Kravatz, D., Diagonal Approximations on an  $n$ -gon and the Cohomology Ring of Closed Compact Orientable Surfaces. Senior Thesis. Millersville University Department of Mathematics, 2008. <http://www.millersville.edu/~rumble/StudentProjects/Kravatz/finaldraft.pdf>
- [12] Latecki L. J., 3D Well-Composed Pictures. Graphical Models and Image Processing 59 (3) (1997), 164-172.
- [13] Munkres J.R., Elements of Algebraic Topology. Addison-Wesley Co., 1984.
- [14] Peltier S., Ion A., Kropatsch W.G., Damiand G., Haxhimusa Y., Directly Computing the Generators of Image Homology Using Graph Pyramids. Image and Vision Computing 27 (7) (2009), 846-853.
- [15] Serre J.P., Homologie Singuliere des Espaces Fibres, applications. Ann. Math. 54 (1951) 429-501

DEPT. OF APPLIED MATH (I), SCHOOL OF COMPUTER ENGINEERING, UNIVERSITY OF SEVILLE,  
CAMPUS REINA MERCEDES, C.P. 41012, SEVILLE, SPAIN

*E-mail address:* `rogodi@us.es`

PATTERN RECOGNITION DEPARTMENT, ADVANCED TECHNOLOGIES APPLICATION CENTER, 7TH  
AVENUE #21812 218 AND 222, SIBONEY, PLAYA, C.P. 12200, HAVANA CITY, CUBA

*E-mail address:* `jlar@cenatav.co.cu`

DEPARTMENT OF MATHEMATICS, MILLERSVILLE UNIVERSITY OF PENNSYLVANIA, P.O. Box 1002  
MILLERSVILLE, PA 17551-0302, PENNSYLVANIA, USA

*E-mail address:* `ron.umble@millersville.edu`